Project 8: Planetary orbits

Optional Project.

This is an optional project which can contribute to your over all project grades.

It is due 9th of Novermber at 4:59.

Submissions should be made via the link on the course website. Please submit a single zip file containing your own code and a single typed PDF document containing all figures, answers to any questions and any extra work you have done.

You are welcome to work together but you should hand in your own work.

Project briefing

This week's project starts with an incomplete program to simulate the orbital trajectory of a planet around the Sun. Your first task is to complete the program by including the equations of motion in Heun form. You will then use the completed program to verify Kepler's laws and (optionally) explore a hypothetical non-Newtonian law of gravity.

Part 1.

The program `orbitx.py` uses the Heun Method to compute the orbital trajectory for a small mass (m — the planet) around a large mass (M —the Sun). Since $M >> m$, we may assume that the Sun remains fixed in place, while the planet orbits around it. Ignoring all forces other than the Sun's gravity, we may write the equations of motion for the planet as

$$m\frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^3}\mathbf{r} \quad \text{and} \quad \frac{d\mathbf{r}}{dt} = \mathbf{v},$$

where **v** is the velocity of the planet and **r** the radius vector from the Sun to the planet. G is the universal constant of gravitation.

Your first step should be to write down the set of differential equations for the components of the vectors **r** and **v** in cartesian (x,y) co-ordinates. Once you have done this, recast them in Euler-Cromer form.

The natural units for this problem are the Astronomical Unit (AU; 1 AU = 1.496×10^{11} m) and the year (1 yr = 3.15×10^7 s). What is the value of the product GM in these units?

Download `orbitx.py` from the course website. Examine the code carefully. You should notice that some crucial lines are missing – in particular the lines that evaluate the acceleration, velocity and position of the planet. When complete, the program should

- compute and plot the orbital trajectory (r versus θ) for a given number of time steps, `Nsteps`;
- compute and plot the kinetic, potential and total energy (per unit mass) of the planet at each time step;
- compute and plot the magnitude of the orbital angular momentum (per unit mass).

The program will prompt the user for the following input parameters:

`v0` the initial tangential velocity at perihelion (in AU/yr) `r0` the initial radial distance at perihelion (in AU)

`Nsteps` total number of time steps

`dt` time step (yr)

Completing the program

To get the program to work you will need to:

a) Complete the statements which evaluate the kinetic and potential energies and the angular momentum (remember that the angular momentum involves the vector product **r×v**).

b) Insert lines that evaluate the *x*- and *y*-components of the acceleration, velocity and radius vectors, using the Euler-Cromer method.

Test the modified code. Does it give sensible results? Briefly describe what you did to test the program and record the results.

Part 2.

When you are satisfied that the completed program is working properly, use it to investigate the following situations. For investigations a) and b), keep `r0` fixed at a value of 1 AU.

a) – Circular orbit. Set `v0` to a value that produces a circular orbit (don't use trial and error — work it out). Run simulations for values of `dt` = 0.001, 0.01, 0.1. In each case, set `Nsteps` so that the planet completes at least 3 orbits. What happens to the energy and angular momentum as you increase the time step? What do you think should happen? How does the percentage change in total energy vary with the time step? What is the largest value of `dt` that gives a "good" simulation?

b) – Elliptical orbits. Modify the program slightly so that it finds the **eccentricity** of the orbit and prints it to the screen. Change `v0` so that you get a mildly elliptical orbit and run simulations for the same values of `dt` as in the previous case. What happens to the energy and angular momentum now? Briefly explain what you observe.

Now explore the effect of varying `v0`. Run the program for several values of `v0`, and record the eccentricity and total energy for each case. Is there a relationship between these quantities? Is it what you would expect? Comment on the results.

c) – Simulating Planetary orbits. Modify the program slightly so that it finds the **period of the orbit** and writes it to the screen. The table below lists the semi-major axis (a) and eccentricity (e) for the orbits of several planets. Use these data to determine the appropriate values of `r0` and `v0` for each planet (in natural units) and run the program for a least one complete orbit (you will probably need to adjust `dt` and `Nsteps`). In a table, record the semi-major axis (as given) and the orbital period and total energy as returned by the program. Comment on the results – are they physically reasonable (e.g., is the total energy close to the theoretical value)? Plot a log-log graph of period (in years) versus semi-major axis (in AU). Does your program reproduce Kepler's 3^{rd} Law?

To receive full credit you must submit the following on paper:

a) Derivation of equations of motion in Heun form.

b) Test results and comments.

c) Results for part 2a), illustrated with *selected* graphs of orbital trajectories; energy versus time and angular momentum versus time.

d) Results for part 2b), illustrated as above.

e) Graphs of orbital trajectory for each planet listed in part 2c), together with a graph of semi-major axis versus period.

Planet	**Semi-major axis (AU)**	**Eccentricity**
Mercury	0.387	0.206
Earth	1.000	0.017
Mars	1.524	0.093
Jupiter	5.203	0.048
Saturn	9.54	0.056
Pluto	39.44	0.249

Further investigation (optional)

Suppose that you live in an alternative universe where gravity follows an inverse cube law rather than an inverse square law (i.e. gravitational force $F = G'Mm/r^3$). Modify your program accordingly and use it to explore the consequences. Assuming that $G'M = 4\pi^2$ (in units of AU and yr), can you find a value of `v0` that yields a circular orbit? What happens if you make a slight change in this value?