Euler's Method for 2nd order ODE

Given the second order differential equation,

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\omega^2\boldsymbol{x} \tag{1}$$

where ω has solution $\boldsymbol{x}(t) \in \mathbb{R}^N$. We can rewrite (1) as 2 first order differential equations.Let,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v},$$

then

$$\frac{d\boldsymbol{v}}{dt} = -\omega^2\boldsymbol{x}.$$

where $\boldsymbol{v} \in \mathbb{R}^N$. If $N = 1$ we can write this as two coupled differential equations in one dimension. In vector form we have

$$\begin{bmatrix} \frac{dx}{dt} \\ \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ \\ -\omega^2 x \end{bmatrix} \tag{2}$$

Euler's Method

Euler's Method can be written for the ODE,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{F}(\boldsymbol{x}, t), \quad \text{for} \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{3}$$

as

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + h\boldsymbol{F}(\boldsymbol{x}_i, t_i)$$

where h is the step size, $\boldsymbol{x}_i \in \mathbb{R}^N$ is the solution to (3) at time t_i

We can apply Euler's method to (2) and get,

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ v_i \end{bmatrix} + h \begin{bmatrix} v_i \\ -\omega^2 x_i \end{bmatrix}. \tag{4}$$

See SHM.py on the project 5 web page for an implementation of this where $\omega = 1$.