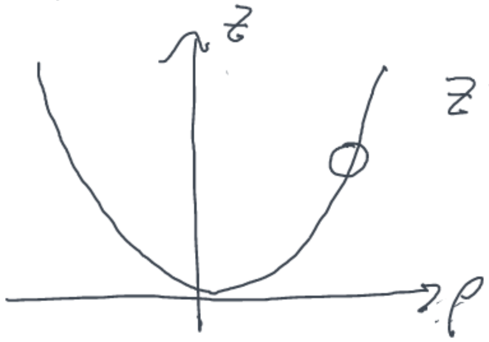
Homework #6

#1



$z = k\rho^2$ $\qquad U = mgz$

$T = \frac{1}{2}mv^2$

$= \frac{1}{2}m(\dot\rho^2 + (\rho\dot\phi)^2 + \dot z^2)$

$= \frac{1}{2}m(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2)$

$$L = \frac{1}{2}m(\dot\rho^2 + \rho^2\omega^2 + 4k^2\rho^2\dot\rho^2) - mgk\rho^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\rho} = \frac{d}{dt}\left(m\dot\rho + 4mk^2\rho^2\dot\rho\right) = m\ddot\rho + 4mk^2(2\rho)\dot\rho^2 + 4mk^2\rho^2\ddot\rho$$

$$\frac{\partial L}{\partial\rho} = m\rho\omega^2 + m4k^2\rho\dot\rho^2 - 2mgk\rho$$

$$m\ddot\rho + 8mk^2\rho\dot\rho^2 + 4mk^2\rho^2\ddot\rho = m\rho\omega^2 + m4k^2\rho\dot\rho^2 + 2mgk\rho$$

(*) $(1 + 4k^2\rho^2)\ddot\rho + 4k^2\rho\dot\rho^2 = (\omega^2 - 2gk)\rho$

ρ_0 is equilibrium if $\dot\rho = 0$ and $\ddot\rho = 0$

from (*) iff $(\omega^2 - 2gk)\rho = 0$

<u>Case 1</u> $\rho = 0$ $\qquad \dot\rho = 0$

for small perturbation $\ddot\rho \approx (\omega^2 - 2gk)\rho$

iff $\omega^2 < 2gk$ $\rho = 0$ is stable

If $\omega^2 > 2gk$ unstable.

Case 2 $\omega^2 = 2gk$ $\forall \rho$

give small nudge in either direction for any ρ. From ✱

Bead will move away.

(2a) $dt = r\, d\tau$ $\ddot{x} + \mu \frac{x}{r^3} = 0$

$$\dot{x} = \frac{d}{dt} x = \frac{1}{r} x'$$

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\left(\frac{1}{r}x'\right) = \frac{1}{r}\frac{d}{d\tau}\left(\frac{1}{r}x'\right) = \frac{1}{r}\left(\frac{-r'}{r^2}x' + \frac{1}{r}x''\right)$$

$$\frac{1}{r}\left(\frac{1}{r}x'' - \frac{r'x'}{r^2}\right) = \frac{-\mu x}{r^3}$$

$$r x'' - r'x' + \mu x = 0 \quad \text{(✱)}$$

(b) $\frac{1}{2}|\dot{x}|^2 - \frac{\mu}{r} = -h$ $h = \frac{\mu}{2a}$ $h > 0$ elliptic

$$\frac{1}{2}\left|\frac{1}{r}x'\right|^2 - \frac{\mu}{r} = -h$$

$$\frac{1}{2r^2}|x'|^2 - \frac{\mu}{r} = -h$$

(c) $x = u^2$ $x' = 2uu'$ $x'' = 2u'^2 + 2uu''$

$$r = u\bar{u}$$

$$r' = u'\bar{u} + u\bar{u}'$$

$$|u|^2(2u'^2 + 2uu'') - (u'\bar{u} + u\bar{u}')2uu' + \eta u^2 = 0$$

$$2r\mu u'' + \cancel{2r\mu u'^2} - \cancel{2u'\bar{u}\mu u'} - 2u\bar{u}'\mu u' + \eta u^4 = 0$$

(*) becomes

$$2ru'' + \eta u - 2|u'|^2 u = 0 \quad (!)$$

Energy becomes $2(u')^2 = \eta - rh$

Thus (!) becomes $u'' + \omega^2 u = 0$

$$\omega = \sqrt{\frac{h}{2}}$$

#3 (a) from class notes $r = a$ $\ddot{r} = 0$ $x = r - a$

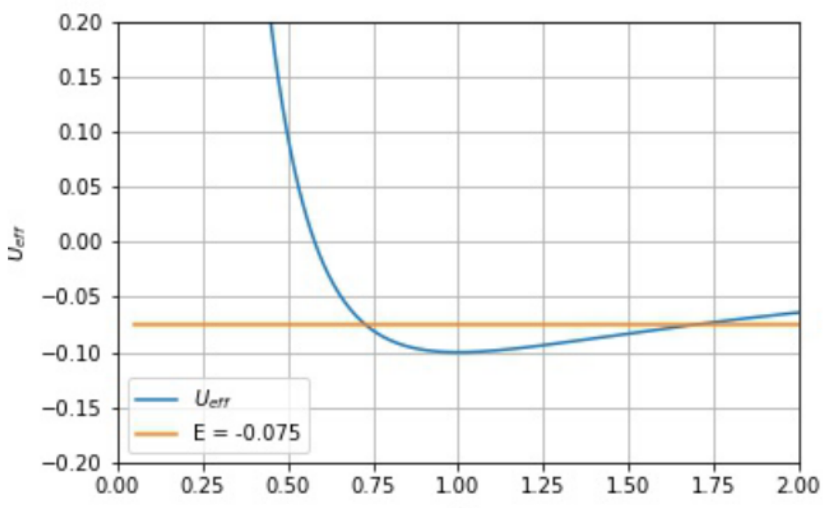
Stable if $F(a) + \frac{a}{3}f'(a) < 0$

$$F(r) = \frac{-K}{r^3} \qquad f'(r) = \frac{3K}{r^4}$$

$$F(a) + \frac{a}{3}f'(a) = -\frac{K}{a^3} + \frac{a}{3}\left(\frac{3K}{a^4}\right) = 0$$

since $f(a) + \frac{a}{3}f'(a)$ not < 0 orbit is unstable

(b)



b) $r_0 = 1$

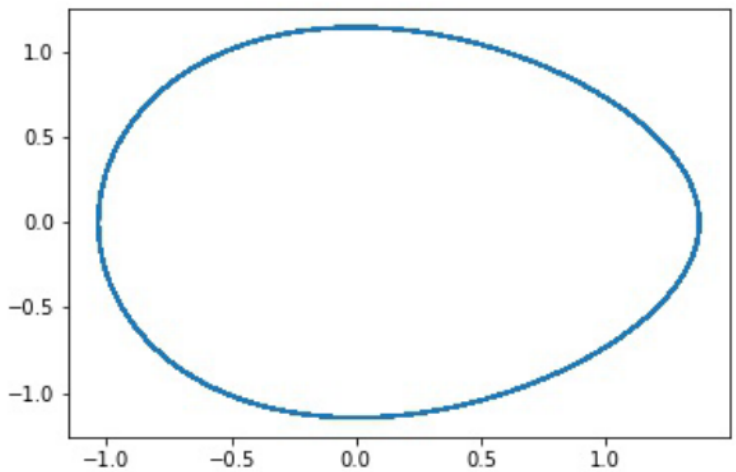
c) $r_{min} = 0.7279697$

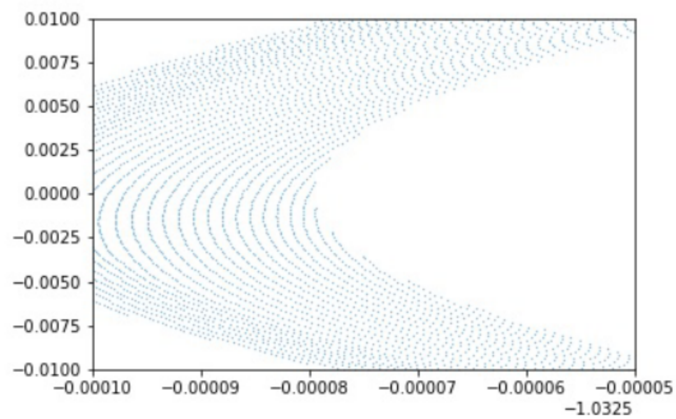
(d) $F = \frac{-k}{r^{8/3}}$ $F = -k r^{-8/3}$ $r = \frac{1}{u}$

$= -k\left(\frac{1}{u}\right)^{-8/3} = -k\, u^{8/3}$

$$u'' = -u - \frac{\mu}{l^2 u^2}\left(-k u^{8/3}\right)$$

$$u'' = -u + \frac{\mu k}{l^2} u^{2/3}$$





Not periodic.

8.20/ a) $c = \frac{l^2}{\gamma \mu}$ $l \to 0$ $c \to 0$

$r_{max} = \frac{c}{1-\varepsilon}$ if r_{max} to be fixed $\varepsilon \to 1$

$2a = r_{min} + r_{max}$ so $a \longrightarrow r_{max}/2$

perihelion gets close to sun.

b) for $l \to 0$ $l \neq 0$ $a = \frac{r_{max}}{2}$ Thus $\tau_{(l\to 0)} = \frac{\pi\, r_{max}^{3/2}}{\sqrt{2GM_s}}$

c) when $l = 0$ $E = \frac{1}{2} m v^2 - \frac{\gamma}{r} = \frac{-\gamma}{r_{max}}$

so $v_r = -\sqrt{2GM_s}\sqrt{1/r - 1/r_{max}}$

time to fall

$$t = \int_{r_{max}}^{0} \frac{dr}{v_r} = \int \frac{dr}{\sqrt{2GM_s}\sqrt{1/r - 1/r_{max}}} = \frac{\pi\, r_{max}^{3/2}}{2\sqrt{2GM_s}}$$

d) period $4t$

e) Twice as long

8.11 $\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - \frac{1}{2} k \vec{r}^2 \Rightarrow \mu \ddot{\vec{r}} = -k\vec{r}$

Soln $x = A\cos\omega t + B\sin\omega t$

$y = C\cos\omega t + D\sin\omega t$

$$Cx - Ay = (BC - DA)\sin\omega t$$

$$Dx - By = -(BC - DA)\cos\omega t$$

sqr and add $\quad a = C^2 + D^2 \quad b = -(CA + DB)$

$c = A^2 + B^2 \quad K = (BC - DA)^2 > 0$

$$ax^2 + 2bxy + cy^2 = K$$

$a, c > 0 \quad ac > b^2 \quad ac - b^2 > 0$

Thus r moves around ellipse.